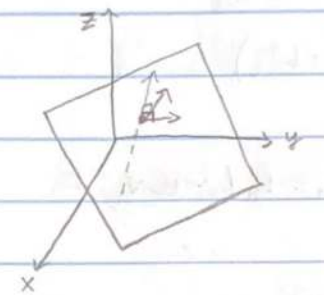
09/01/21

12.4 Cross Product

*NOTE: Everything today lives in $\mathbb{R}^3$

Goal: Given two vectors, construct a third (nonzero if possible) vector, orthogonal to both.



Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

Suppose $\vec{w} = \langle w_1, w_2, w_3 \rangle$ is the desired vector, what must $\vec{w}$ satisfy?

Because $\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$:

$$\begin{cases} 0 = \vec{w}\cdot\vec{u} = w_1u_1 + w_2u_2 + w_3u_3 & (1) \\ 0 = \vec{w}\cdot\vec{v} = w_1v_1 + w_2v_2 + w_3v_3 & (2) \end{cases}$$

Multiply (1) by v_3 and (2) by u_3 to obtain:

$$\begin{cases} 0 = v_3(\vec{w}\cdot\vec{u}) = w_1(u_1v_3) + w_2(u_2v_3) + w_3(u_3v_3) & \text{Eq } 1^* \\ 0 = u_3(\vec{w}\cdot\vec{v}) = w_1(v_1u_3) + w_2(v_2u_3) + w_3(u_3v_3) & \text{Eq. } 2^* \end{cases}$$

Aside: $-ax+by=0$ has a solution $x=b$, $y=a$ $\leadsto -ab+ba=0$

Subtracting 2^* from 1^*:

$$0 = v_3(\vec{w}\cdot\vec{u}) - u_3(\vec{w}\cdot\vec{v})$$
$$= w_1(u_1v_3 - u_3v_1) + w_2(u_2v_3 - u_3v_2)$$
$$= -w_1(-(u_1v_3 - u_3v_1)) + w_2(u_2v_3 - u_3v_2)$$

Hence we have a solution:

$$\begin{cases} w_1 = u_2v_3 - u_3v_2 \\ w_2 = -(u_1v_3 - u_3v_1) \end{cases}$$

Now we plug back into 1 to obtain ω_3.

$$
\begin{aligned}
0 &= \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \\
&= (u_2v_3 - u_3v_2)u_1 + (-(u_1v_3 - u_3v_1))u_2 + \omega_3 u_3 \\
&= u_1u_2v_3 - u_1u_3v_2 - u_1u_2v_3 + u_2u_3v_1 + \omega_3 u_3 \\
&= (u_2v_1 - u_1v_2)u_3 + \omega_3 u_3 \\
&= u_3(u_2v_1 - u_1v_2 + \omega_3)
\end{aligned}
$$

So, either $u_3 = 0$ or $\omega_3 = u_1v_2 - u_2v_1$

$$
\begin{aligned}
\vec{\omega} &= \langle \omega_1, \omega_2, \omega_3 \rangle \\
&= \langle u_2v_3 - u_3v_2,\ -(u_1v_3 - u_3v_1),\ u_1v_2 - u_2v_1 \rangle
\end{aligned}
$$

Now we can check this to verify that $\vec{\omega}$ satisfies:

$$
\begin{cases} \vec{\omega} \cdot \vec{u} = 0 \\ \vec{\omega} \cdot \vec{v} = 0 \end{cases}
$$

Determinants

DEFINITION: The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

The determinant of a 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$

is: $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$= a(ek - fh) - b(dk - fg) + c(dh - eg)$$

Ex. Find the determinant of $\begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det \begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 1 \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - (-2) \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= 1\left(-1(1) - 0(1)\right) - (-2)\left(-1(1) - 0(1)\right) + 3\left(-1(0) - (-1)(0)\right)$$

$$= 1(-1) + 2(-1) + 3(0)$$

$$= -1 - 2 + 0 = \boxed{-3}$$

<u>Turns Out</u>: That vector is a symbolic determinant

$$\begin{array}{r} \\ \vec{u} \rightarrow \\ \vec{v} \rightarrow \end{array} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \hat{\imath}\left(u_2 v_3 - u_3 v_2\right) - \hat{\jmath}\left(u_1 v_3 - u_3 v_1\right) + \hat{k}\left(u_1 v_2 - u_2 v_1\right)$$

$$= \left\langle u_2 v_3 - u_3 v_2,\ -(u_1 v_3 - u_3 v_1),\ u_1 v_2 - u_2 v_1 \right\rangle$$

This is the same $\vec{w}$ we computed before!

DEFINITION: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle, \vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$.

The cross product of $\vec{u}$ with $\vec{v}$ is

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

NOTE: The cross product as an operation takes two vectors in $\mathbb{R}^3$ and creats another vector in $\mathbb{R}^3$

Properties of Cross Product (Algebraic)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
5. Also geometric → $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Properties of Cross Product (Geometric)

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$

1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
2. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$ (θ is angle between $\vec{u}$ & $\vec{v}$)
3. $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u} \times \vec{v} = 0$